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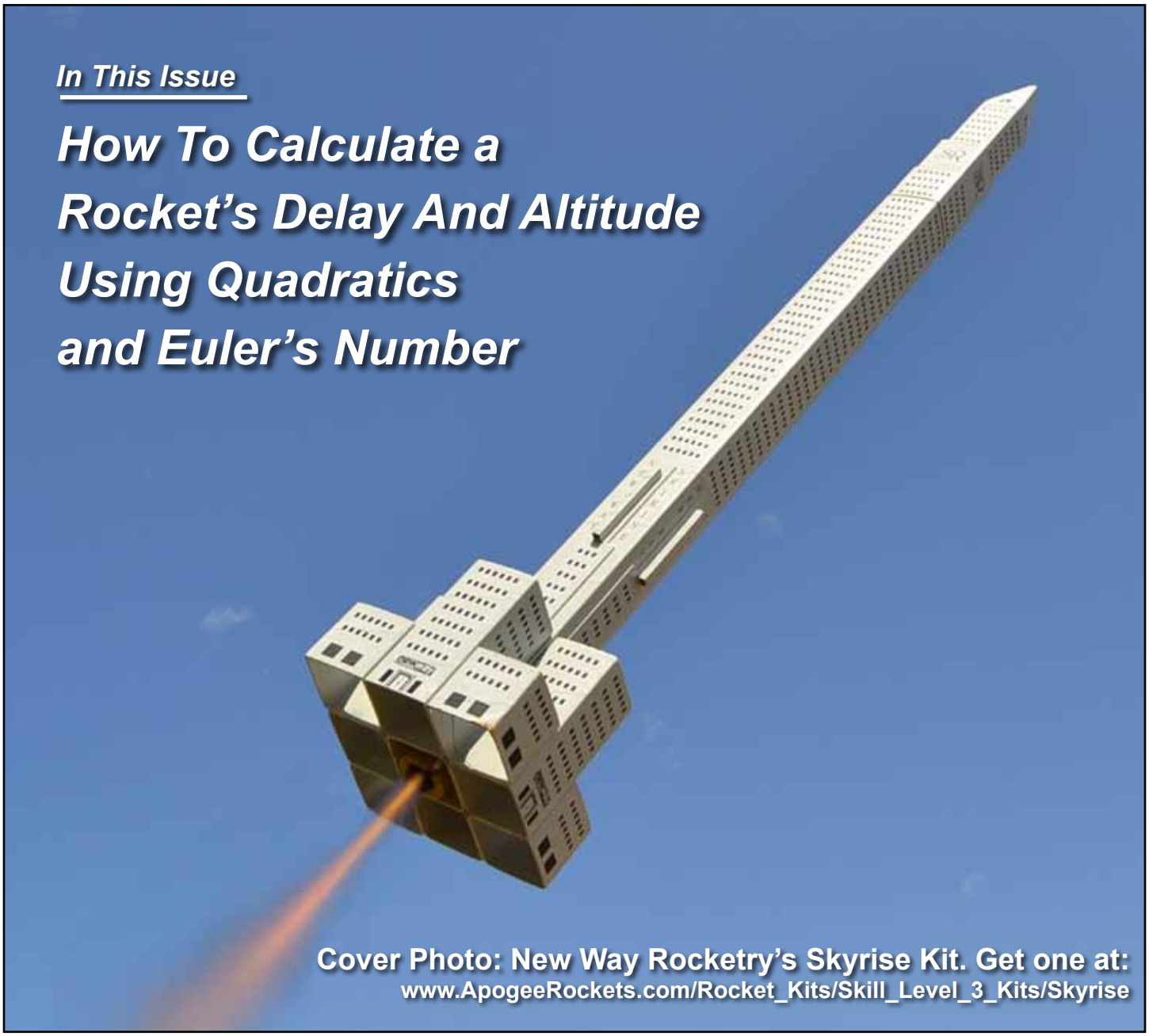
NEWSLETTER

In This Issue

How To Calculate a Rocket's Delay And Altitude Using Quadratics and Euler's Number



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Calculating the Delay Period Using Quadratics and Euler's Number

By Ian Voss

{Editor's Note: Before there was computer software that made our lives so easy, modelers had to calculate by hand the flight parameters of the rocket. This article shows you how to do just that. It is not likely that you'll do these formulas by hand, but you'll find this to be a good leap into rocketry eduction. I'm sure a few teachers will like their students to walk through this mathematical derivation.

Also, our author is from Australia, so you may notice some words spelled a little differently than we spell them here in the United States of America.}

Simulation software is great for designing model rockets and determining how they will perform. I use and recommend them, especially if you are scratch-building. It takes the guesswork out of selecting a delay period (and a great many among other things) so that you can avoid high speed deployment events and potential damage to the rocket.

On occasion though, you may find yourself without access to a computer (or the required software) wanting to test a previously untried combination of rocket and motor. For those in this situation, it's possible to calculate the delay period with reasonable accuracy using only a pen and paper (although, having a calculator will definitely make the process easier). If you don't use simulation software (I can't imagine why you wouldn't want to), these calculations can be set up on a spreadsheet. Before you get too excited though, there are some provisos:

1. This method does not account for dynamic stability and assumes that you are launching vertically in dead calm conditions. It does however, allow for aerodynamic drag.

2. Because we are allowing for drag, you will need to either know or take a guess at the C_D (Drag Co-efficient). You will also need to know the maximum airframe diameter. You could calculate the projected frontal cross-sectional area including the fins and launch lug but it's really not necessary. The fins and launch lug have a very small frontal area compared to the airframe and our 'averaged' C_D is only approximate so it is more practical to just base our calculations on the airframe diameter.

3. You will need to know the Total Impulse and the Thrust Duration of the motor you intend to use. This information should be supplied with the documentation that comes with the motor. Or, you could take values from the motor manufacturer's catalogue.

4. You will need to know the weight of the rocket, i.e. the weight of the rocket when it is fully prepped and ready to launch. You could allow for propellant mass and use the average weight of the rocket but I won't be doing this in my example. It would unnecessarily complicate what is only an approximation anyway.

5. This method is fine for low powered rockets. I've compared the values obtained from this method with what comes out of a simulator for several small rockets and they're well within 10%. I have not tested for and would not expect accuracy for higher speeds and altitudes. Perhaps the reader could test the accuracy with their own measurements?

6. If your rocket has apogee detection, the delay period is irrelevant. You might however, want to use this method to determine the altitude at apogee.

7. Metric units will be used (metres, kilograms, Newtons) so if you are not familiar with these, now would be a good time to get familiar with them. With metric units, conversion factors can be avoided which simplifies the process. If you prefer to measure in feet, it's easy enough to convert the results.

8. I would expect that finding the way through this process would be daunting for some. I would recommend taking a notepad with a template already in place if you intend to use this method at the range.

An Overview Of The Process

Here is an outline of the process. Armed with the above information, we'll form a *quadratic equation* which will give

Continued on page 3

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Continued from page 2

Calculating the Coast Delay Period

us the maximum velocity (V_{max}) of the rocket (or a close approximation at least). Once we know V_{max} , we can form another quadratic (or something that closely resembles one) and use that to determine the delay period. There's some calculus along the way (integration) but I'll avoid the usual notation and instead try to make it look like a template where you can just plug in the values. As a bonus, once the delay period is known, a few more simple calculations will give us the altitude at apogee.

The critical value is C_D and one would do well to keep in mind that the value we choose may not be a true value as might be obtained from wind tunnel testing. Rather, for the purpose of these calculations, it is more like an average value. Remember that we are not allowing for dynamic stability or weather conditions so what may be correct for one flight may give erratic values with different motors and/or different conditions. When using different motors, the static margin will change and this will have an effect on the dynamic stability. As the dynamic stability changes, the overall effect of aerodynamic drag is likely to change.

To allow for dynamic stability and weather conditions, the only practical solution would be to use simulation software which brings us back to the reason for this article. In short, this method is aimed at providing an approximation of how a model rocket will perform with a given motor when you don't have access to a computer. The best we can hope for are 'ballpark' figures – not as good as a simulation but still better than a complete guess.

Calculating the Maximum Velocity

Rocket motors are designed to produce their maximum thrust as quickly as possible so that the rocket reaches a stable speed before it leaves the launch rod. While this is good in terms of providing a stable flight, it makes it rather complicated to analyse what is happening to the rocket over a series of short time increments. As the thrust changes, so does the acceleration. The mass of the rocket is also changing as the propellant is burnt. The only constant is gravity! Luckily for us, initially we won't be looking for any value other than V_{max} and we can find this by making some assumptions.

Step 1 is to ignore the motor's thrust curve entirely and simply assume that the thrust is constant throughout the thrust phase. This simplifies the calculations considerably which is what we want for a pen and paper (and calculator) exercise. In my example, I'm going to use an Estes 'Alpha' fitted with a A8-3 motor and a lift-off weight of 50 grams.

For the A8-3 motor, the Estes catalogue specifies a total impulse of 2.50 Newton.seconds (Ns) and a thrust duration of 0.5 seconds.

Total Impulse: $I_{Total} = 2.50Ns$

Thrust Duration: $t_1 = 0.5s$

The *average thrust* for the ½ second interval is:

$F_T = I_{Total} / t_1$

$= 2.50 / 0.5$

$= 5N$ We'll need this value soon.

Step 2 is to find the *weight* of the rocket. This is easily determined using Newton's 2nd Law (F = ma). A common misconception is that the unit for weight is kilograms. From a *strictly scientific perspective*, weight is the result of the effect of gravity and is a force and force is measured in Newtons. Note that we will define everything in the upward direction as having a positive value and everything in the downward direction as having a negative value.

Continued on page 4

Continued from page 3

Calculating the Coast Delay Period

The mass of our rocket is: $m = 0.050kg$

The acceleration due to gravity is: $a_g = -9.8\ m/s^2$

Therefore, the weight of the rocket is:

$F_W = (0.050kg)(-9.8m/s^2)$

$= -0.5N$ We'll also need this value.

It's interesting to note that even the acceleration due to gravity is not constant but reduces in value as you get further from Earth's surface. Over the range of altitudes we are dealing with however, the change is negligible and so we can assume that it is a constant.

Step 3 is to define an equation which will allow for aerodynamic drag. We need an equation because drag is not constant. The last two values that we've found can be assumed to be constant. With drag however, the value is dependent on the velocity of the rocket. In fact, it is dependent on the square of the velocity – if the velocity is doubled, then the drag increases by a factor of four. The equation that we define will form the last term in equation 2 which will lead us to our quadratic (equation 4).

To allow for aerodynamic drag, we'll calculate the Force of drag:

$F_D = ½\ \rho_{air} A_{CS} C_d v^2$

$= (-1.6 \times 10\text{-}4)\ v^2$ **equation 1**

Where:

$\rho_{air} = 1.225\ kg/m^3$ (Which is the average density of air at sea level.)

(For the purpose of these calculations, ρ_{air} is assumed to be a constant.)

$A_{CS} = 4.8 \times 10^{-4}\ m^2$ -- This is the cross-sectional area of the body tube at it's maximum diameter. The Estes 'Alpha' uses BT50 body tube which has a diameter of 24.8mm.

$C_d = 0.550$ This is an estimate of the rocket's drag co-efficient. Remember that this is more like an average value for our calculations.

$v_1 = ?$ This is the velocity of the rocket which is as yet undetermined.

Step 4 is to define the *net force* which is applied to the rocket during the thrust phase. This is where our calculations up till now, come together.

$F_{net} = F_T + F_W + F_D$

$= 5 - 0.5 - (1.6 \times 10^{-4})\ v^2$

$= 4.5 - (1.6 \times 10^{-4})\ v^2$ **equation 2**

It's interesting to note that although the motor is producing 5N of thrust, only 4.5N of force goes towards accelerating the rocket. To put it a different way, the rocket will not go anywhere until the thrust is greater than the weight of the rocket. Also, when v = 0 the last term cancels out. Now that we've defined the net force, we can define the acceleration.

The *acceleration during the thrust phase* is given by:

$a_1 = Fnet / m$ $(= \Delta v / \Delta t)$

$= [4.5 - (1.6 \times 10^{-4})v^2]/0.05$

$= 90 - (3.2 \times 10^{-3})v^2$ **equation 3**

Now, $\Delta v / \Delta t$ = (the change in velocity) / (the change in time). Normally, Δt is a very short time interval but we can apply it here.

Since we started at $v = 0$ and $t = 0$, we will let Δv = the velocity at the end of the thrust duration (which is approxi-

Continued on page 5

Continued from page 4

Calculating the Coast Delay Period

mately equal to V_{max}) and let Δt = the thrust duration (t_1).

We need to transpose (re-arrange) the equation for acceleration:

$90 - (3.2 \times 10^{-3})\, v^2 = \Delta v/\Delta t$

$90 - (3.2 \times 10^{-3}) v^2 = \Delta v/0.5$

$\Delta v = 45 - (1.6 \times 10^{-3}) v^2$

We will let $\Delta v = v$ and transpose to give:

$(1.6 \times 10^{-3}) v^2 + v - 45 = 0$ **equation 4**

Step 5 involves using 'the quadratic equation' (for lack of a better name). The beauty of it is that for any quadratic (any equation that contains one variable for which two is the highest exponent), you can find a solution. This is known as finding the 'roots' of the equation.

Equation 4 is such an equation. That is, equation 4 is a quadratic of the general form:

$av^2 + bv + c$

where:

a = (1.6 x 10-3)

b = 1

c = -45

We may find v thus:

$v = [\, -b +/- \sqrt{(b^2 - 4ac)}\,]/\, 2a$

$= [-1 +/- \sqrt{(1 - 4 \times (1.6 \times 10^{-3}) \times -45)}]/[2 \times (1.6 \times 10^{-3})]$

$= [-1 +/- \sqrt{(1 + 0.288)}]/(3.2 \times 10^{-3})$

$= [-1 +/- 1.135\,]/(3.2 \times 10^{-3})$

We'll now reject the negative case, giving:

$v = [-1 + 1.135\,]/(3.2 \times 10^{-3})$

We have found V_{max} = 42.2 m/s

A couple more simple calculations and we will know the altitude at which the thrust period ends.

The average acceleration of the rocket over the ½ second interval is found thus:

$a = \Delta v/\Delta t = 42.2/0.5 = 84.4 m/s^2$ (about 8½ g's)

We can determine how far the rocket has travelled in that ½ second like this:

$s_1 = ½\, a\, t^2 = (½)(84.4)(0.5)^2 = 10.5m$ (~34feet)

To summarise so far, the rocket has accelerated from zero to approximately 42m/s (140 ft/s) in about half a second and reached an altitude of 10.5m (35ft). If you can imagine a video of the rocket's flight, where we've hit the 'pause' button at burnout we would see the rocket at about 10.5 metres above its launch point.

Calculating the Delay Period

The rocket's maximum velocity (42.2m/s) coincides (approximately) with motor burnout and the motor's delay element is now burning. The question now is, for how long will the rocket coast before it reaches apogee? To simplify the remaining equations, we will assume that at the beginning of the coast phase t = 0.

The acceleration due to gravity is constant and is:

$a_g = -9.8\ m/s^2$ (= a_{final})

The initial acceleration due to drag will be:

$a_d = (-3.2 \times 10^{-3})(42.2)^2$

$= -5.7\ m/s^2$

The net initial acceleration during the coast phase will be:

Continued on page 6

Continued from page 5

Calculating the Coast Delay Period

$a_g + a_d = -9.8 + -5.7$

$= -15.5 \text{ m/s}^2 \ (= a_{initial})$

Note that I say *initial acceleration* and that it is negative. Both gravity and aerodynamic drag are slowing the rocket. As the rocket slows, drag will have less of an effect and gravity will eventually be the only force acting on the rocket. When the rocket falls and is accelerated towards the ground, the force exerted by aerodynamic drag will become positive but we are not concerned with that part of the flight. Hopefully the parachute will deploy and prevent our rocket from reaching a high enough speed to cause damage to either the rocket or anything on the ground.

Using Euler's Number

In the title of this article I mentioned *Euler's number* and you may be wondering now what it has to do with rockets. Euler's number is represented by the symbol 'e' and is approximately equal to 2.72. It is a value that is useful in explaining growth and decay cycles in biology, sociology, economics and physics where the rate of change is proportional to the value. The nice thing about e is that when used in an equation, the graph that results is a smooth curve (a curve not unlike the acceleration curve which we see when a rocket is subject to a negative acceleration during the coast phase). By incorporating e in the equations below, we can approximate what is happening to the rocket as a result of gravity and aerodynamic drag.

The *acceleration with respect to time* is given by:

$a_c = a_{final} + (a_{initial} - a_{final})e^{-t}$

$= -9.8 + (-15.5 + 9.8)e^{-t}$

$= -9.8 - 5.7e^{-t}$ **equation 5**

Note that when $t = 0$ equation 5 equals the net initial acceleration. This is why we 're-set the clock to zero' at the beginning of the coast phase.

Here is where the calculus comes in. If we have an equation for acceleration, we can obtain an equation for velocity by *integrating* the equation for acceleration. Don't worry if you don't know how to integrate an equation, that's been done for you below and there are some notes at the end of the article on how to do it if you can't see the pattern here.

Integrating equation 5 will give the *velocity with respect to time.*

$v = -9.8t + 5.7\,e^{-t} + (42.2 - 5.7)$

$= -9.8t + 5.7\,e^{-t} + 36.5$ equation 6

Note at a constant is added (in brackets) so that when $t = 0$, $v = 42.2$.

Integrating again gives the displacement (s) with respect to time. Again, a constant is added so that when $t = 0$, $s = 0$

$s = -4.9t^2 - 5.7\,e^{-t} + 36.5t + (5.7)$

$= -4.9t^2 + 36.5t - 5.7\,e^{-t} + 5.7$ **equation 7**

Equation 7 resembles a quadratic. I say resembles because the 3rd term has an exponent of –t.

As t increases, the 3rd term ($5.7e^{-t}$) will tend towards zero so we will ignore this term and treat equation 7 as though it is a quadratic. This is 'fudging' the results a little, but we are only concerned with an approximation so it is a fair assumption. The advantage of this is that we can apply the usual rules that apply to quadratics.

Since the 'a' term in equation 7 is negative, s will have a *maximum value* when $t = -b/2a$.

Continued on page 7

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Continued from page 6

Calculating the Coast Delay Period

The delay period will be approximately:

t_{delay} = -(36.5) / 2(-4.9) = 3.7 seconds

Substituting 3.7s in equation 7 gives the altitude gained during the coast phase:

$s_2 = -4.9(3.7)^2 + 36.5(3.7) - 5.7\,e^{-(3.7)} + 5.7$

= 73.5 m (241 feet)

Recall that the altitude at the end of the thrust phase was: s_1 = 10.5 m

The altitude at apogee will be approximately:

$s_1 + s_2$ = 10.5 + 73.5

= 84 m (275 feet)

A crash course in integration

If you're struggling to see how equations 6 and 7 were obtained, here's some more detail:

Term 1 of equation 6 is: -9.8t

Which could be written as: $-9.8t^1$ (Any number to the power of one equals itself)

When we integrate, we increase the exponent by 1 and divide the co-efficient by the new exponent. Term 1 becomes: $[-9.8/(1+1)]t^{(1+1)} = -4.9t^2$

Term 2 of equation 6 is: $5.7\,e^{-t}$

To integrate an e term, neither e nor its exponent is changed.

The coefficient is divided by the exponent's coefficient: $(5.7)/(-1)\,e^{(-1)(t)} = -5.7e^{-t}$

About the Author

Ian Voss is a member of the Queensland Rocketry Society (QRS - www.qldrocketry.com) in Australia. He enjoys attending the regular club launch days and designing and scratch-building rockets in his spare time. He is currently the holder of an altitude record (for an A8-5 motor) and looks forward to gaining his L1 certification in the near future. Ian firmly believes in the benefits of getting involved in the hobby of model rocketry. *"As a tool to promote learning, rocketry is hard to beat. It's exciting and fun and captures the imagination like few other things can."*

Ian and his son at a fete display to promote the QRS and rocketry in general. The rockets in the background belong to Ari Piirainen (far left) who is the Public Relations man for the club.